ENEE322: Signals and Systems, Fall 2009
Complementary Notes for *Signals and Systems* by Alan V. Oppenheim and Alan S. Willsky, Second Edition

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Chapter 1

Signals and Systems

1.1 Continuous-Time and Discrete-Time Signals

- A **signal** is simply a function of one or more independent variables.
- A **continuous-time** signal $x(t)$ is a function of a real number, t.
- A **discrete-time** signal, or sequence, $x[n]$ is a function of an integer, n. Each element of a discrete-time sequence is called a **sample**.
- The **energy** and **power** in a continuous-time signal is

  $$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt,$$

  and in discrete time,

  $$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

1.2 Transformations of the Independent Variable

- **Time shift**: Let $t_0 > 0$. If $y(t) = x(t - t_0)$, then $y(t)$ is a delayed copy of $x(t)$, i.e., $x(t)$ is shifted to the right by t_0 along the horizontal axis.

  If $y(t) = x(t + t_0)$, then $y(t)$ is an advanced copy of $x(t)$, i.e., $x(t)$ is shifted to the left by t_0 along the horizontal axis.
- **Time scaling**: Let $a > 0$. If $y(t) = x(at)$ where $a < 1$, then $y(t)$ is $x(t)$ stretched along the horizontal axis.

  If $a > 1$, then $y(t)$ is $x(t)$ compressed along the horizontal axis.
- **Time reversal**: If $y(t) = x(-t)$, then $y(t)$ is $x(t)$ reflected about the vertical axis.

1.3 Exponential and Sinusoidal Signals

- A **general complex exponential** takes the form $x(t) = Ce^{at}$ where both C and a are in the set of complex numbers.
- A **periodic complex exponential** takes the form $x(t) = e^{j\omega_0 t}$, i.e., C is real and a is imaginary. This signal is periodic with fundamental frequency ω_0 and fundamental period $T_0 = 2\pi/\omega_0$. Note that $|e^{j\omega_0 t}| = 1$ for all t.

1.4 The Unit Impulse and Unit Step Functions

- In discrete time, the **unit impulse function** and the **unit step function** are defined as

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}.$$

- The unit impulse and unit step functions satisfy the following properties:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad \text{and} \qquad \delta[n] = u[n] - u[n-1].$$

- In continuous time, the **unit step function** is

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

- The **unit impulse function**, $\delta(t)$, can be interpreted as a limit of the function $\delta_\Delta(t)$ as $\Delta \to 0$, where

$$\delta_\Delta(t) = \begin{cases} 1/\Delta & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases}.$$

  As Δ decreases, the square pulse $\delta_\Delta(t)$ because narrower and taller, yet its integral remains equal to 1. Therefore, in the limit,

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}.$$

  Formally, $\delta(t)$ is not a proper function over the domain of real numbers because $\delta(0)$ "equals" infinity. Nevertheless, for our purposes, we will continue to manipulate $\delta(t)$ as if it were an ordinary function, for example, by scaling, shifting, integrating, and differentiating.

- The unit impulse, $\delta(t)$, satisfies the following properties:

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1, \qquad \int_{-\infty}^{t} \delta(\tau) d\tau = u(t), \qquad \delta(t) = \frac{du(t)}{dt},$$

$$\text{and} \qquad x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

1.5 Continuous-Time and Discrete-Time Systems

- A **system** is any unit that takes an input signal and yields and output signal.
- We will use the notation $x(t) \to y(t)$ to denote a system with input $x(t)$ and output $y(t)$.

1.6 Basic System Properties

- Suppose that there is a system such that $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$. This system is **linear** if and only if
    1. for any constant a, $ax_1(t) \to ay_1(t)$, and
    2. $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$.
- A system is **time-invariant** if and only if $x(t - t_0) \to y(t - t_0)$, i.e., a shifted input yields a shifted output with the same amount of shift.
- A system is **memoryless** if and only if the output at time t_0, $y(t_0)$, is only a function of $x(t_0)$ and not a function of $x(t)$ for any $t \neq t_0$. In other words, the present output depends only on the present input.
- A system is **invertible** if and only if distinct inputs lead to distinct outputs, i.e., if $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, then $y_1(t) = y_2(t)$ for all $t \Rightarrow x_1(t) = x_2(t)$ for all t.
- A system is **causal** if and only if the output at time t_0, $y(t_0)$, is only a function of $x(t)$ for $t \leq t_0$ and not a function of $x(t)$ for any $t > t_0$. In other words, the present output depends only on present and past inputs.

Chapter 2

Linear Time-Invariant Systems

2.1 Discrete-Time LTI Systems: The Convolution Sum

- The **sifting** property illustrates how a discrete-time signal $x[n]$ can be represented as a sum of unit impulses, scaled and shifted appropriately.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- The **impulse response**, $h[n]$, of an LTI system is defined to be the output of the system when the input is the unit impulse, $\delta[n]$.

- We will use the notation $x[n] \xrightarrow{LTI} y[n]$ to denote an LTI system with input $x[n]$ and output $y[n]$.

- By time invariance, if $\delta[n] \xrightarrow{LTI} h[n]$, then $\delta[n-n_0] \xrightarrow{LTI} h[n-n_0]$. Then by linearity, $A\delta[n-n_1] + B\delta[n-n_2] \xrightarrow{LTI} Ah[n-n_1] + Bh[n-n_2]$. Therefore, by linearity and time invariance,

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xrightarrow{LTI} \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

  By the sifting property, the left-hand side is equal to $x[n]$. Therefore, when the input to an LTI system is $x[n]$, the output is equal to the **convolution** between $x[n]$ and the impulse response, $h[n]$:

$$\boxed{y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].}$$

2.2 Continuous-Time LTI Systems: The Convolution Integral

- The **sifting** property illustrates how a continuous-time signal $x(t)$ can be represented as an integral of unit impulses, scaled and shifted appropriately.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

- The **impulse response**, $h(t)$, of an LTI system is defined to be the output of the system when the input is the unit impulse, $\delta(t)$.
- By time invariance, if $\delta(t) \xrightarrow{LTI} h(t)$, then $\delta(t-t_0) \xrightarrow{LTI} h(t-t_0)$. Then, by linearity, $A\delta(t-t_1) + B\delta(t-t_2) \xrightarrow{LTI} Ah(t-t_1) + Bh(t-t_2)$. Therefore, by linearity and time invariance,

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \xrightarrow{LTI} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

  By the sifting property, the left-hand side is equal to $x(t)$. Therefore, when the input to an LTI system is $x(t)$, the output, $y(t)$, is equal to the **convolution** between $x(t)$ and the impulse response, $h(t)$:

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.}$$

2.3 Properties of LTI Systems

- In both continuous time and discrete time, convolution is **commutative**, **distributive**, and **associative**:

Property	Example in continuous time
commutative	$x(t) * h(t) = h(t) * x(t)$
distributive	$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$
associative	$(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$

Table 2.1: Basic properties of convolution.

- An LTI system is **memoryless** if and only if its impulse response takes the form $h(t) = K\delta(t)$ where K is any constant, i.e., the output is simply a scaled copy of the input.
- If an LTI system with impulse response $h(t)$ is **invertible**, then there exists a signal $h_I(t)$ such that $h(t) * h_I(t) = \delta(t)$, i.e., $h_I(t)$ is the impulse response of the inverse system.
- An LTI system is **causal** if and only if $h(t) = 0$ for all $t < 0$.
- An LTI system is **stable** if and only if $\int_{-\infty}^{\infty} |h(t)|dt < \infty$, i.e., the impulse response is absolutely integrable (or summable).
- The **step response**, $s(t)$, of an LTI system is defined to be the output of the system when the input is the unit step, $u(t)$.
- All of the above properties hold for both continuous time and discrete time.

2.4 Causal LTI Systems Described by Differential and Difference Equations

- A general N^{th}-order **linear constant-coefficient differential equation** takes the form

  $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

  where $a_N \neq 0$. This differential equation describes a system with input $x(t)$ and output $y(t)$.

- The condition of **initial rest** of a differential equation states that, if $x(t) = 0$ for $t \leq t_0$, then $y(t) = 0$ for $t \leq t_0$. If a differential equation has the condition of initial rest, then it describes a causal and LTI system.

- A general N^{th}-order **linear constant-coefficient difference equation** takes the form

  $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

  where $a_N \neq 0$. This difference equation describes a system with input $x[n]$ and output $y[n]$.

- The condition of **initial rest** of a difference equation states that, if $x[n] = 0$ for $n \leq n_0$, then $y[n] = 0$ for $n \leq n_0$. If a difference equation has the condition of initial rest, then it describes a causal and LTI system.

2.5 Singularity Functions

- Using convolution, we can restate the sifting property for continuous-time signals as

  $$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t).$$

  This property can be used to *define* the unit impulse as the function $\delta(t)$ which, for any signal $x(t)$, $x(t) = x(t) * \delta(t)$.

- We already know that the integral $\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$. Therefore, $\delta(t) = u'(t)$.

- The "derivative" of $\delta(t)$, $\delta'(t)$, can be defined in a limiting sense as $\delta'_\Delta(t)$ as $\Delta \to 0$, where

  $$\delta'_\Delta(t) = \frac{d}{dt}\frac{1}{\Delta}\left(u(t) - u(t-\Delta)\right) = \frac{1}{\Delta}\left(\delta(t) - \delta(t-\Delta)\right)$$

  Observe that

  $$x(t) * \delta'_\Delta(t) = \int_{-\infty}^{\infty} x(\tau)\left(\frac{1}{\Delta}\left(\delta(t-\tau) - \delta(t-\tau-\Delta)\right)\right)d\tau = \frac{x(t) - x(t-\Delta)}{\Delta}.$$

  By the definition of the derivative, as $\Delta \to 0$, $x(t) * \delta'_\Delta(t)$ approaches $x'(t)$.

  Therefore, the function $\delta'(t) = \lim_{\Delta\to 0} \delta'_\Delta(t)$ has the property that

  $$\boxed{x(t) * \delta'(t) = x'(t).}$$

- Also, $\delta'(t) * u(t) = u'(t) = \delta(t)$.
- In continuous time, $\delta(t)$, $u(t)$, and $\delta'(t)$ belong to a class of functions known as **singularity functions**.

Chapter 3

Fourier Series Representation of Periodic Signals

3.2 Response of LTI Systems to Complex Exponentials

- When $x(t) = e^{st}$ is the input to an LTI system and $h(t)$ is its impulse response, then the output is

  $$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) e^{st} \ .$$

  Similarly, when $x[n] = z^n$ is the input to an LTI system and $h[n]$ is its impulse response, then the output is

  $$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z) z^n \ .$$

- $H(s)$ does not depend on t, and $H(z)$ does not depend on n. Therefore, for these choices of inputs, the output is simply a scaled version of the input. A signal input for which the system output is simply a scaled version of the input is called an *eigenfunction*, and the scaling factor is called the *eigenvalue.*

- By linearity, when $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$ is the input to an LTI system, then the output is $y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$.

- **Conclusion**: When the input can be expressed as a weighted sum of complex exponentials, i.e., $x(t) = \sum_k a_k e^{s_k t}$, then computing the output $y(t)$ *does not require convolution.* That property is desirable because convolution can sometimes be difficult.

- *When* can the input be expressed in this manner? We will show that when $x(t)$ is periodic, $x(t)$ can be expressed as a weighted sum of complex exponentials.

3.3 The Continuous-Time Fourier Series

- A signal $x(t)$ is **periodic** with period T if $x(t) = x(t+T)$ for all t. The smallest positive number T for which this holds is called the fundamental period, and $\omega_0 = 2\pi/T$ is the fundamental frequency.

- The **synthesis equation** for the **continuous-time Fourier series** for the periodic signal $x(t)$ with period T is

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

and the **analysis equation** is

$$\boxed{a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt}.$$

The values a_k are often called the **Fourier series coefficients**.

3.4 Convergence of the CT Fourier Series

- Given the choice of a_k shown earlier, can any periodic signal $x(t)$ be expressed as this weighted sum of complex exponentials? In other words, does

$$\lim_{K\to\infty} \sum_{k=-K}^{K} a_k e^{jk\omega_0 t} = x(t)?$$

- If the following **Dirichlet conditions** hold, then yes, $x(t)$ has a Fourier series expansion.

  1. $x(t)$ must be absolutely integrable.
  2. $x(t)$ must have a finite number of oscillations within a single period.
  3. $x(t)$ must have a finite number of discontinuities within a single period.

3.5 Properties of the CT Fourier Series

- **Multiplication**: The Fourier series coefficients of a product of two CT signals is equal to the convolution of the corresponding FS coefficients of the two signals.

$$\left.\begin{array}{c} x(t) \leftrightarrow a_k \\ y(t) \leftrightarrow b_k \end{array}\right\} \Rightarrow z(t) = x(t)y(t) \leftrightarrow c_k = \sum_{m=-\infty}^{\infty} a_m b_{k-m}$$

- **Parseval's Relation**: The energy in one period of the signal $x(t)$ is proportional to the energy in the sequence a_k.

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

3.6 The Discrete-Time Fourier Series

- A signal $x[n]$ is **periodic** with period N if $x[n] = x[n+N]$ for all n. The smallest positive integer N for which this holds is called the fundamental period, and $\omega_0 = 2\pi/N$ is the fundamental frequency.
- The **synthesis equation** for the **discrete-time Fourier series** for the periodic signal $x[n]$ with period N is

  $$\boxed{x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}}$$

  and the **analysis equation** is

  $$\boxed{a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n}}$$

  where the notation $\sum_{k=<N>}$ is equivalent to $\sum_{k=m}^{m+N-1}$ for your choice of integer m, i.e., to sum over any N consecutive integers.
- In continuous time, any periodic signal $x(t)$ with fundamental frequency $\omega_0 = 2\pi/T$ could be represented as a weighted sum of harmonically-related complex exponentials $\{e^{jk\omega_0 t}\}|_{k=-\infty}^{\infty}$. There is an infinite number of harmonically-related complex exponentials that are periodic with period T.
- In discrete time, there is only a *finite* number of harmonically-related complex exponentials that are periodic with period N because, for any integer r,

  $$e^{j(k+rN)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} e^{j2\pi rn} = e^{jk\frac{2\pi}{N}n}.$$

  Therefore, the synthesis equation for the discrete-time Fourier series only requires N terms at most. Equivalently, the sequence of FS coefficients a_k is **periodic** with period N.

3.7 Properties of the DT Fourier Series

- **Multiplication**: The Fourier series coefficients of a product of two DT signals is equal to the periodic convolution of the corresponding FS coefficients of the two signals.

  $$\left.\begin{array}{l} x[n] \leftrightarrow a_k \\ y[n] \leftrightarrow b_k \end{array}\right\} \Rightarrow z[n] = x[n]y[n] \leftrightarrow d_k = \sum_{m=<N>} a_m b_{k-m}$$

- **Parseval's Relation**: The energy in one period of the signal $x[n]$ is proportional to the energy in one period of the sequence a_k.

  $$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

3.8 Fourier Series and LTI Systems

- When $x(t) = e^{st}$ is the input to an LTI system and $h(t)$ is its impulse response, then the output is
$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)e^{st} .$$
Similarly, when $x[n] = z^n$ is the input to an LTI system and $h[n]$ is its impulse response, then the output is
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n .$$
- The functions $H(s)$ and $H(z)$ are called **system functions** or **transfer functions** of the corrresponding systems. Note that $H(s)$ does not depend on t, and $H(z)$ does not depend on n. Therefore, for these choices of inputs, the output is simply a scaled version of the input.
- When $s = j\omega$, $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$ is called the **frequency response** of the system. When $z = e^{j\omega}$, $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ is the frequency response of the system. Note how $H(j\omega)$ and $H(e^{j\omega})$ are only functions of ω.
- When $x(t) = e^{jk\omega_0 t}$ is the input to an LTI system, then the output is $y(t) = H(jk\omega_0)e^{jk\omega_0 t}$. By linearity, when $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ is the input to an LTI system, then the output is $y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$.
- **Conclusion**: If $\{a_k\}$ is the set of Fourier series coefficients for $x(t)$, then $\{a_k H(jk\omega_0)\}$ is the set of Fourier series coefficients for $y(t)$. If $\{a_k\}$ is the set of Fourier series coefficients for $x[n]$, then $\{a_k H(e^{jk\omega_0})\}$ is the set of Fourier series coefficients for $y[n]$.

3.9 Filtering

- Passing a signal through an LTI system is often referred to as **filtering** or frequency-selective filtering.

Filter Type	Passband	Stopband
low-pass	low frequencies	high frequencies
high-pass	high frequencies	low frequencies
band-pass	middle frequencies	low and high frequencies
band-stop	low and high frequencies	middle frequencies

Table 3.1: Properties of common frequency-selective filters.

3.10 CT Filters Described by Differential Equations

- Assuming initial rest, consider the causal LTI system described by the following differential equation:
$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

When $x(t) = e^{j\omega t}$, we already know that $y(t) = H(j\omega)e^{j\omega t}$. Therefore,

$$j\omega H(j\omega)e^{j\omega t} + aH(j\omega)e^{j\omega t} = e^{j\omega t}$$

which yields

$$H(j\omega) = \frac{1}{a + j\omega}.$$

The magnitude response (squared) of this filter is

$$|H(j\omega)|^2 = \frac{1}{a + j\omega}\frac{1}{a - j\omega} = \frac{1}{a^2 + \omega^2}.$$

3.11 DT Filters Described by Difference Equations

- Assuming initial rest, consider the causal LTI system described by the following difference equation:

  $$y[n] = ay[n-1] + x[n] \ .$$

  When $x[n] = e^{j\omega n}$, we already know that $y[n] = H(e^{j\omega})e^{j\omega n}$. Therefore,

  $$H(e^{j\omega})e^{j\omega n} = H(e^{j\omega})e^{j\omega(n-1)} + e^{j\omega n}$$

  which yields

  $$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}.$$

  The magnitude response (squared) of this filter is

  $$|H(e^{j\omega})|^2 = \frac{1}{1 - ae^{-j\omega}}\frac{1}{1 - ae^{j\omega}} = \frac{1}{1 - 2a\cos\omega + a^2}.$$

- Consider the first difference:

  $$y[n] = x[n] - x[n-1].$$

  The impulse response of this system is

  $$h[n] = \delta[n] - \delta[n-1].$$

  The frequency response of this system is

  $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = 1 - e^{-j\omega}.$$

  The magnitude response (squared) of this filter is

  $$|H(e^{j\omega})|^2 = (1 - e^{-j\omega})(1 - e^{j\omega}) = 2 - 2\cos\omega.$$

Chapter 4

The Continuous-Time Fourier Transform

4.1 Representation of Aperiodic Signals

- The Fourier series is able to express (most) periodic signals as a sum of complex exponentials, where the sequence of FS coefficients $\{a_k\}$ completely characterize the signal. Does there exist a similar representation for aperiodic signals?
- Motivating example: Consider the periodic signal $x(t)$ with period T:

  $$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2. \end{cases}$$

  The FS coefficients for this signal are

  $$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k} = \frac{\sin(2\pi k(T_1/T))}{\pi k}$$

  Draw a_k as a function of ω where $\omega = k\omega_0$. When T increases, the spacing of a_k decreases in frequency. Equivalently, its frequency resolution increases.

  As T approaches infinity, the sequence of peaks will approach its continuous envelope. This envelope, a continuous function of frequency, is the Fourier transform of $x(t)$.
- Suppose $x(t)$ is an aperiodic signal. For now, assume $x(t) = 0$ for $|t| > T_1$.

  Let $\tilde{x}(t)$ be periodic with period T such that $\tilde{x}(t) = x(t)$ for $|t| \le T_1$. For now, assume $T/2 > T_1$.

  Let a_k be the FS coefficients for the periodic signal $\tilde{x}(t)$. Then,

  $$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}.$$

  Like our earlier definition of a system function from Section 3.8, let us define $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. Then,

$$a_k = \frac{1}{T} X(jk\omega_0).$$

Therefore,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(jk\omega_0) e^{jk\omega_0 t}.$$

As $T \to \infty$, $\tilde{x}(t) \to x(t)$, where $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$.

- The **Fourier transform** of $x(t)$ is

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

and the **inverse Fourier transform** of $X(j\omega)$ is

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega}.$$

- The **Dirichlet conditions** for convergence of the Fourier transform are similar to those required for the Fourier series.

  1. $x(t)$ must be absolutely integrable.
  2. $x(t)$ must have a finite number of oscillations within any finite interval.
  3. $x(t)$ must have a finite number of discontinuities within any finite interval.

- The big picture!

	Continuous Time	Discrete Time
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ continuous and periodic in time $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ discrete and aperiodic in frequency	$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$ discrete and periodic in time $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\frac{2\pi}{N}n}$ discrete and periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ continuous and aperiodic in time $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ continuous and aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ discrete and aperiodic in time $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ continuous and periodic in frequency

Table 4.1: Comparison of Fourier series and Fourier transform.

4.2 Fourier Transform for Periodic Signals

- If $X(j\omega) = 2\pi\delta(\omega - \omega_0)$, then

  $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}.$$

  If $X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)$, then

  $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)e^{j\omega t}d\omega = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}.$$

- Therefore, the Fourier transform of a periodic signal is equivalent to a train of impulses occuring at harmonically related frequencies, where the impulse at $\omega = k\omega_0$ is scaled by $2\pi a_k$.

  $$\boxed{X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)}.$$

4.3 Properties of the CT Fourier Transform

- We will use the notation $x(t) \xrightarrow{FT} X(j\omega)$ to denote $x(t)$ and $X(j\omega)$ as a Fourier transform pair.

- **Linearity**: If $x(t) \xrightarrow{FT} X(j\omega)$ and $y(t) \xrightarrow{FT} Y(j\omega)$, then

  1. for any constant A, $Ax(t) \xrightarrow{FT} AX(j\omega)$, and
  2. $x(t) + y(t) \xrightarrow{FT} X(j\omega) + Y(j\omega)$.

- **Time shift**: If $x(t) \xrightarrow{FT} X(j\omega)$, then $x(t - t_0) \xrightarrow{FT} e^{-j\omega t_0}X(j\omega)$.

- **Time reversal**: If $x(t) \xrightarrow{FT} X(j\omega)$, then $x(-t) \xrightarrow{FT} X(-j\omega)$. Therefore, if $x(t)$ is even, then $X(j\omega)$ is even. If $x(t)$ is odd, then $X(j\omega)$ is odd.

- **Conjugate symmetry**: If $x(t) \xrightarrow{FT} X(j\omega)$, then $x^*(t) \xrightarrow{FT} X^*(-j\omega)$.

  If $x(t)$ is real, i.e., $x(t) = x^*(t)$, then $X(j\omega) = X^*(-j\omega)$.

- **Duality**: For any transform pair, there exists a dual pair with the time and frequency variables interchanged.

- **Parseval's Relation**: If $x(t) \xrightarrow{FT} X(j\omega)$, then

  $$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}.$$

  In other words, the energy in $x(t)$ is proportional to the energy in $X(j\omega)$.