

## Introduction

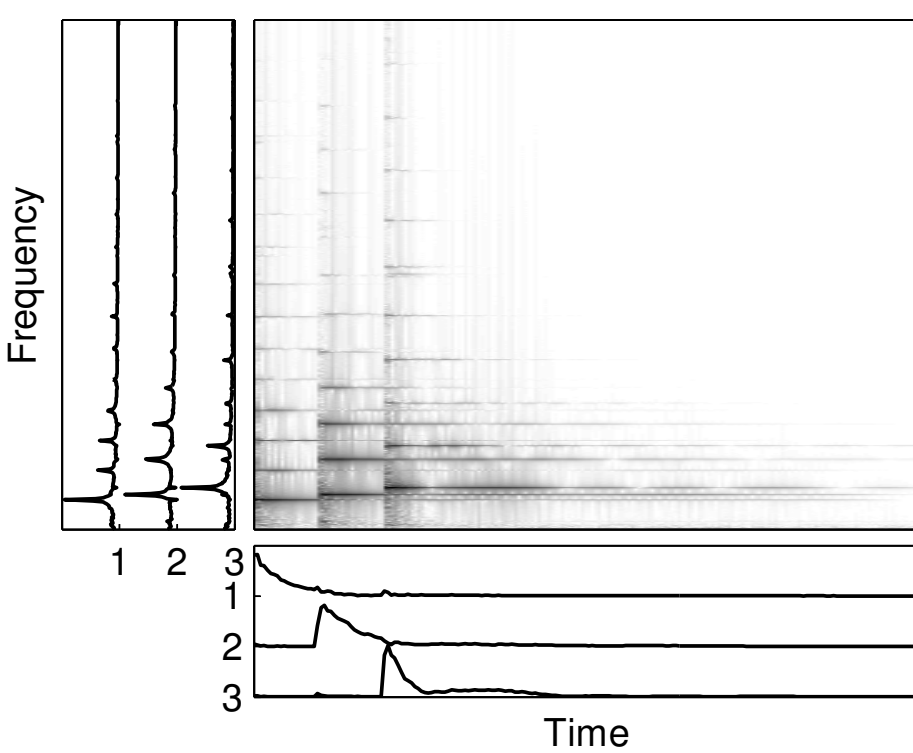
**Nonnegative Matrix Factorization (NMF)** has become a popular tool for decomposing spectrograms of musical signals.

Goal: Given nonnegative  $\mathbf{X}$ , find two nonnegative matrices:

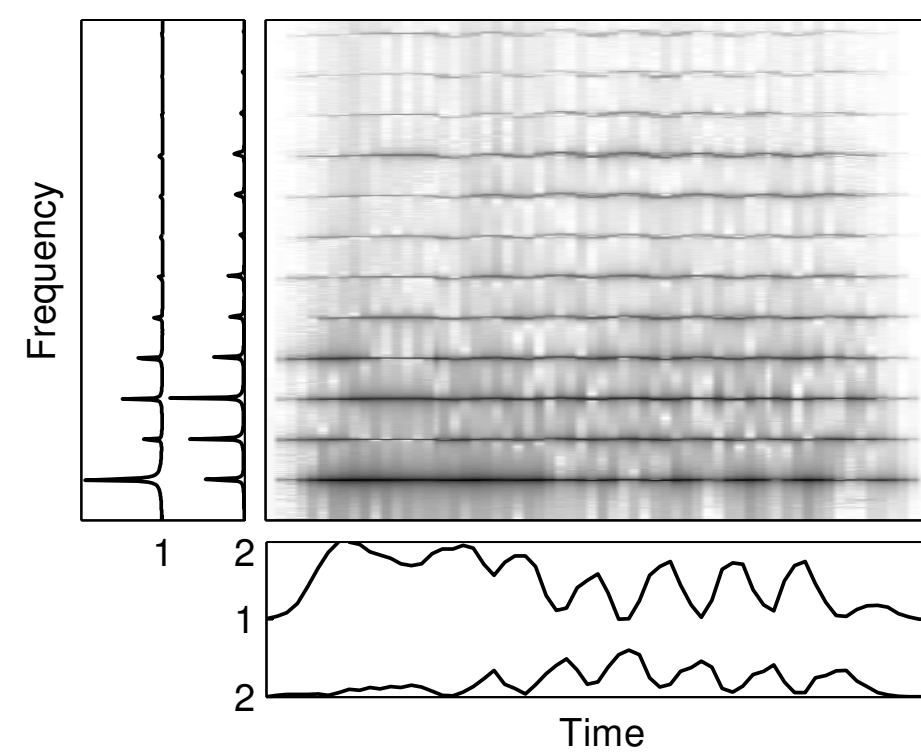
- dictionary:  $\mathbf{A} \in \mathbb{R}_+^{M \times K}$ , where  $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_K]$
- gain matrix:  $\mathbf{S} \in \mathbb{R}_+^{K \times N}$ , where  $\mathbf{S} = [\mathbf{s}_1 \ \mathbf{s}_2 \ \dots \ \mathbf{s}_M]^T$

that minimize some distance between  $\mathbf{X}$  and  $\mathbf{AS}$ .

Piano: three notes



Violin: one note



## Common Multiplicative Update Rules

Euclidean distance:  $d_{\text{EUC}}(x, y) = |x - y|^2$

- $\min_{\mathbf{A}, \mathbf{S}} d_{\text{EUC}}(\mathbf{X}, \mathbf{AS})$ :

$$\mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{XS}^T}{\mathbf{ASS}^T} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\mathbf{A}^T \mathbf{X}}{\mathbf{A}^T \mathbf{AS}}$$

Kullback-Leibler divergence:  $d_{\text{KL}}(x, y) = x \log \frac{x}{y} - x + y$

- $\min_{\mathbf{A}, \mathbf{S}} d_{\text{KL}}(\mathbf{X}, \mathbf{AS})$ :

$$\mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{XS}^T}{\mathbf{AS}^T} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\mathbf{A}^T \mathbf{X}}{\mathbf{A}^T \mathbf{1}}$$

where  $\mathbf{1}$  is a matrix of ones.

Itakura-Saito divergence:  $d_{\text{IS}}(x, y) = \frac{x}{y} - \log \frac{x}{y} - 1$

- $\min_{\mathbf{A}, \mathbf{S}} d_{\text{IS}}(\mathbf{X}, \mathbf{AS})$ :

$$\mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{XS}^T}{\frac{1}{\mathbf{AS}^T}} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\mathbf{A}^T \mathbf{X}}{\mathbf{A}^T \frac{1}{\mathbf{AS}}}$$

## Motivation

Problem: Some musical “objects” require more than one atom for accurate representation.

Solution: Impose **co-occurrence constraints** that force certain atoms to be highly correlated or uncorrelated.

## Problem Formulation

New minimization problem:

$$\min_{\mathbf{A}, \mathbf{S}} d(\mathbf{X}, \mathbf{AS}) + \lambda d(\mathbf{Q}, \mathbf{SS}^T)$$

- $\lambda > 0$  is a regularization parameter,
- $\mathbf{SS}^T$  is a temporal co-occurrence matrix,
- $\mathbf{Q} \in \mathbb{R}_+^{K \times K}$  is a predefined nonnegative symmetric matrix.

How is  $\mathbf{Q}$  chosen?

- $q_{ii} = 1$  for all  $i$  to normalize each row of  $\mathbf{S}$ ,
- $q_{ij} \approx 1$  for all pairs of atoms  $i$  and  $j$  that we want to be dependent,
- $q_{ij} \approx 0$  for all other pairs of atoms.

## Proposed Multiplicative Update Rules

To minimize  $d(\mathbf{Q}, \mathbf{A}^T \mathbf{A})$  or  $d(\mathbf{Q}, \mathbf{SS}^T)$ ,

$$\text{EUC: } \mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{AQ} + \epsilon}{\mathbf{AA}^T \mathbf{A} + \epsilon} \quad \text{or} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\mathbf{QS} + \epsilon}{\mathbf{SS}^T \mathbf{S} + \epsilon}$$

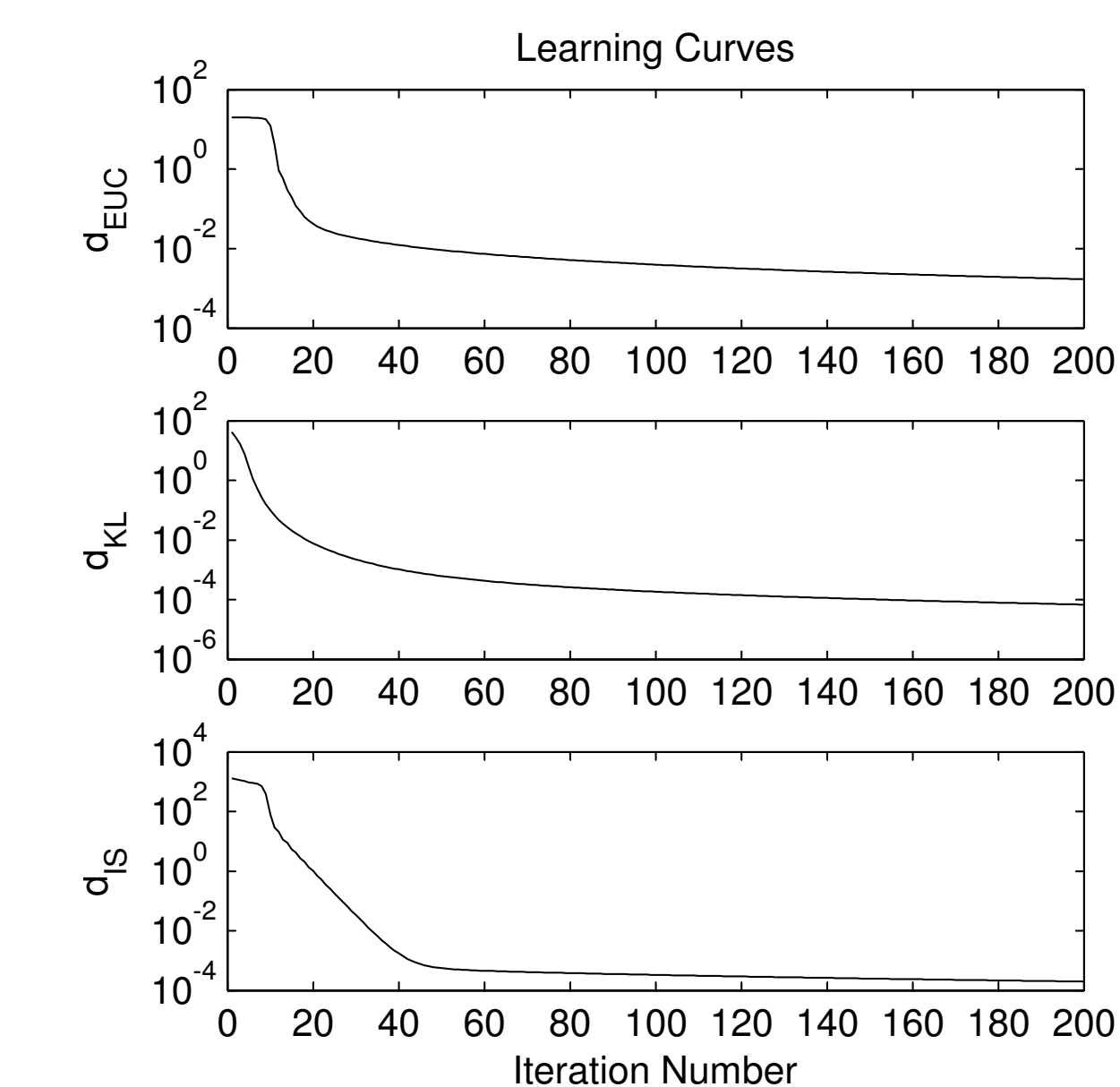
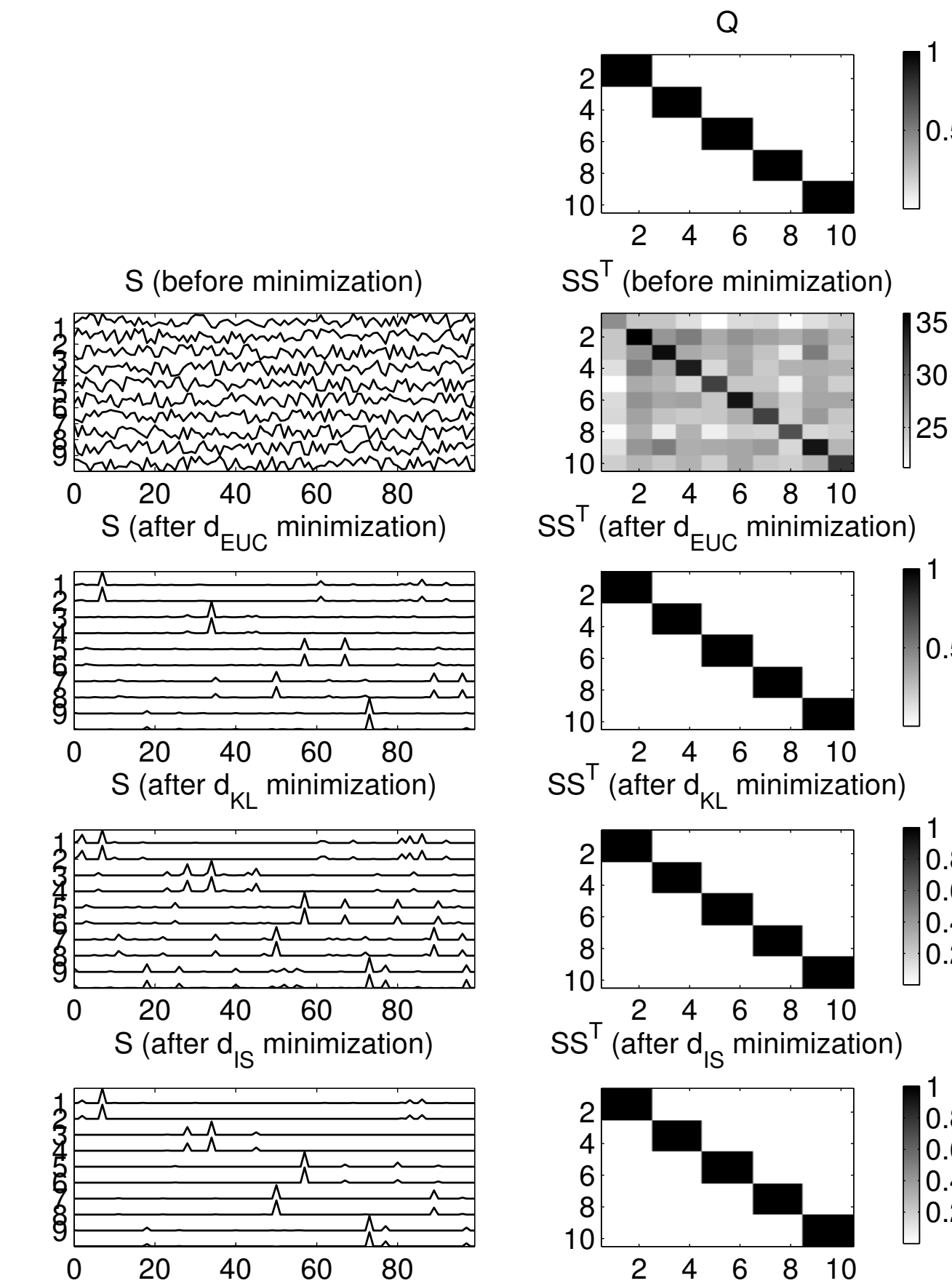
$$\text{KL: } \mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{A} \frac{\mathbf{Q}}{\mathbf{A}^T \mathbf{A}} + \epsilon}{\mathbf{A} \mathbf{1} + \epsilon} \quad \text{or} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\frac{\mathbf{Q}}{\mathbf{SS}^T} \mathbf{S} + \epsilon}{\mathbf{1S} + \epsilon}$$

$$\text{IS: } \mathbf{A} \leftarrow \mathbf{A} \cdot \frac{\mathbf{A} \frac{\mathbf{Q}}{(\mathbf{A}^T \mathbf{A})^2} + \epsilon}{\mathbf{A} \frac{1}{\mathbf{AA}^T} + \epsilon} \quad \text{or} \quad \mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\frac{\mathbf{Q}}{(\mathbf{SS}^T)^2} \mathbf{S} + \epsilon}{\frac{1}{\mathbf{SS}^T} \mathbf{S} + \epsilon}$$

To incorporate into initial NMF problem, the update rule changes to (e.g., for Euclidean distance):

$$\mathbf{S} \leftarrow \mathbf{S} \cdot \frac{\mathbf{A}^T \mathbf{X} + \lambda \mathbf{QS} + \epsilon}{\mathbf{A}^T \mathbf{AS} + \lambda \mathbf{SS}^T \mathbf{S} + \epsilon}$$

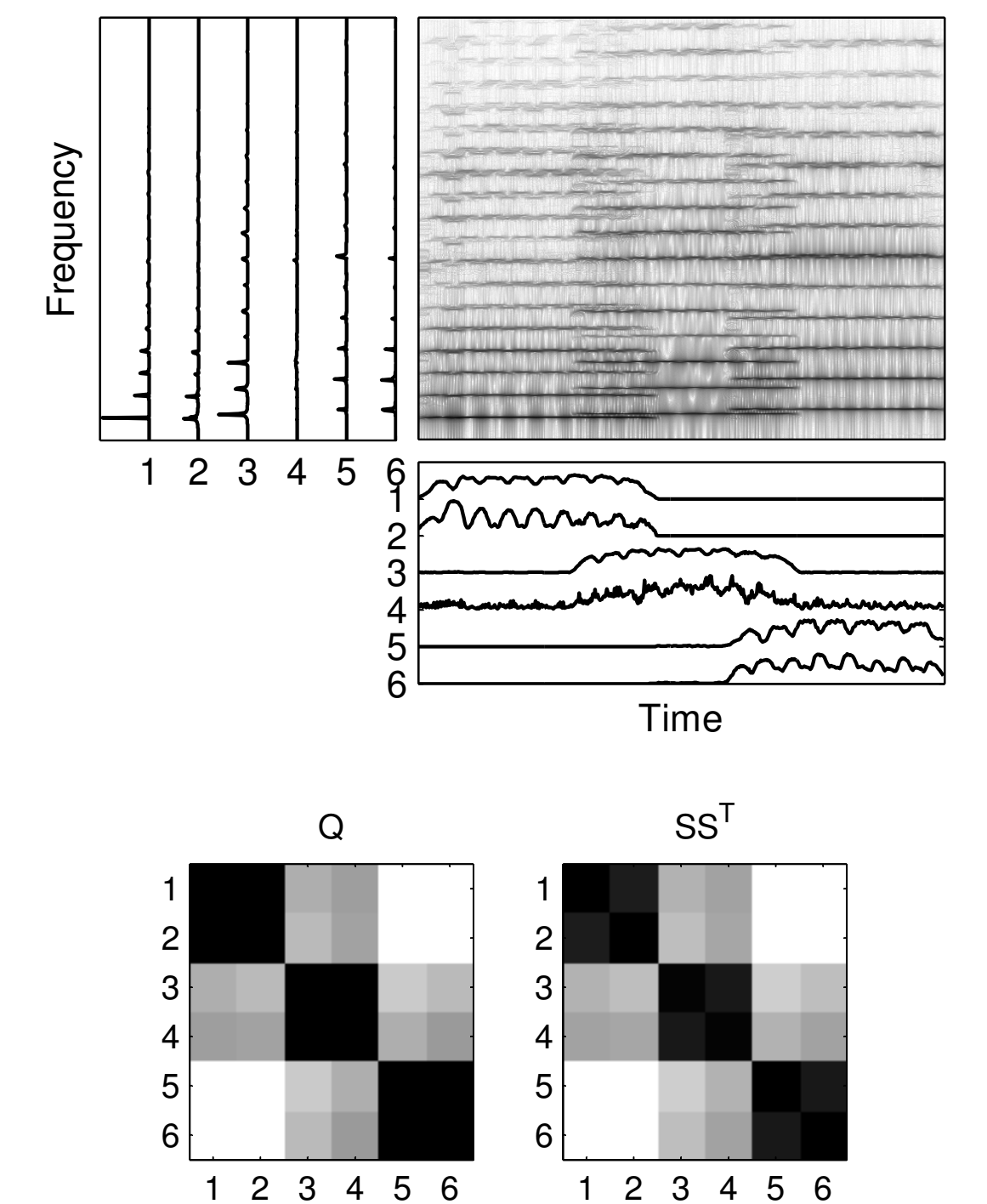
## Experiment: Convergence



## Conclusions

- Co-occurrence constraints successfully enforce dependence upon multiple atoms that describe the same musical object.
- Either temporal or spectral co-occurrence constraints can be enforced successfully.

## Experiment: Violin



## Experiment: Drums

